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# LONG PATH CONNECTIVITY OF REGULAR GRAPHS

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ABSTRACT: Any pair of vertices in a 4-connected non-bipartite k-regular graph are joined by a Hamilton path or a path of length at least 3k-6.

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The topics about Hamilton cycles, circumferences and Hamiltonian connectivities of regular graphs have been interesting many mathematicians in recent years ([2],[1],[4],[7],[3],[6]).

In this paper, we will investigate the length of a longest path joining any pair of vertices of regular graphs and establish the following theorem.

THEOREM 1

Let $G$ be a 4-connected non-bipartite $k$-regular graph. Then any pair of distinct vertices of $G$ are joined by a Hamilton path or a path of length at least $3k-6$.

In a sense, this theorem is a generalization of the rollowing results.

(i) (Bollobas and Hobbs [1]) Any 2-connected $k$-regular graph of order at most $\frac{9}{4}k$ contains a Hamilton cycle.

(ii) (Jackson [4]) Any 2-connected $k$-regular graph of order at most $3k$ contains a Hamilton cycle.

(iii) (Zhu, Liu and Yu [7]) Any 2-connected $k$-regular graph of order at most $3k+3$ contains a Hamilton cycle.

(iv) (Fan [3]) The length of a longest cycle in a 3-connected $k$-regular graph of order $n$ is at least $\min\{n,3k\}$.

(v) (Zhang and Zhu [6]) Any pair of vertices of a 3-connected non-bipartite $k$-regular graph of order at most $3k-4$ are joined by a Hamilton path.

The condition of 4-connectivity in the theorem cannot be reduced. A 3-connected $k$-regular graph of order $3k+3$ containing no path of length at least $2k+3$ joining a pair of vertices can be constructed as follows. Let $k=3h$. Let $G_1,\cdots,G_9$ be nine disjoint copies of complete graph $K_h$ and

$v_1, v_2, v_3$ be three distinct vertices. Join an edge between each pair of vertices in $G_{3i+1}$ $G_{3i+2}$ $G_{3i+3}$ for $i=0,1,2$, and join an edge between $v_j$ and each vertex of $G_{3i+j}$ for $i=0,1,2$ and $j=1,2,3$. The induced graph contains $9h+3$ vertices and is $3h$-regular 3-connected, in which $v_i$ and $v_j$ are not joined by any path of length longer than $6h+2$ for $i,j,\in\{1,2,3\}$. (See fig. 1).

Actually, we can establish a result stronger than Theorem 1.

THEOREM 2. Let $G$ be a 4-connected graph and $x,y$ be a pair of distinct vertices of $G$ such that

(i) $d(v)=k$ for any vertex $v \in V(G)\setminus\{x,y\}$,

(ii) $d(x), d(y) \le k$.

Then the length of a longest path joining $x$ and $y$ is at least

(i) $\min\{|V(G)|-1, 3k-6\}$ if $G$ is not a bipartite graph, or $G$ is a bipartite graph and $x$, $y$ belong to deferent parts of the bipartition of $G$;

(ii) $\min\{|V(G)|-2, 3k-6\}$ if $G$ is a bipartite graph and $x,y$ belong to the same part of the bipartition of $G$.

Let $G=(V,E)$ be a graph with vertex set $V$ and edge set $E$. Let $P=u_0\cdots u_p$ be a path of $G$. For $0 \le i, j \le p$, the segment $u_i \cdots u_j$ of $P$ is denoted by $u_i P u_j$ if $i \le j$ or $u_i \bar{P} u_j$ if $i \ge j$. The length of a path $P$ is the number of edges in $P$ and is denoted by $\ell(P)$. Let $H$ be a subgraph of $G$. Let $w,w'$ be two vertices of $H$. The length of a longest

path of $H$ joining $w,w'$ is denoted by $L_H(w,w')$. Let $v$ be a vertex of $G$. The set of vertices of $H$ adjacent to $v$ is denoted by $N_H(v)$ and the number of vertices of $N_H(v)$ is denoted by $d_H(v)$. When $V(H)=V(G)$, we simply write $d(v)$ and $N(v)$ instead of $d_G(v)$ and $N_G(v)$. Let $P=u_0\cdots u_p$ be a path of $G$ and $X$ be a subset of $V(P)$. Denote

$$X^{+1}=\{u_{i+1}=u_i\in X\}$$

and

$$X^{-1}=\{u_{i-1}=u_i\in X\} .$$

Let $E(H,H')$ be the set of all ordered pairs of vertices $(x,y)$ such that $(x,y)\in E(G)$ and $x\in V(H)$, $y\in V(H')$. And let $|E(H,H')|=e(H,H')$. Note that if $V(H)\cap V(H')\neq\phi$, each edge $(x,y)$ in the induced subgraph $G(V(H)\cap V(H'))$ will counted twice in $e(H,H')$ since the ordered pairs $(x,y)$ and $(y,x)$ are considered deferent in $E(H,H')$. Thus $d(v)=e(v,G)$ for any vertex $v$ of of $G$ and $\sum_{v\in V(H)} d(v)=e(H,G)$ for subgraph $H$ of $G$.

**PROOF OF THEOREM 2**

The theorem will be proved by contradition. Suppose that the length of a longest path $P=v_0\cdots v_p$ joining $x=v_0$ and $y=v_p$ is less than $3k-6$ and $G\setminus V(P)$ is not empty.

**PART ONE.** In this part, we will show that $G\setminus V(P)$ is an independent set of $G$. The following lemmas will be applied in this part.

**LEMMA 1.1.** (Lemma 4, [3]) Let $H$ be a 2-connected graph and $Q=u_0\cdots u_q$ be a longest path of $H$. Then

$$L_H(x,y)\geq \min\{d(u_0),d(u_q)\}$$

*for any pair of distinct vertices* $x$ *and* $y$ *in* $H$.

Let $C$ be a set and $\{A_1,\cdots,A_\alpha\}$, $\{B_1,\cdots,B_h\}$ be partitions of $C$ such that $\alpha \geq 2$ and $|A_\mu \cap B_j| \leq 1$ for any $\mu \in \{1,\cdots,\alpha\}$ and any $j \in \{1,\cdots,h\}$. If

$$B_i \cap A_\mu \neq \phi,\ B_j \cap A_\theta \neq \phi \text{ and } B_{i+1} = \cdots = B_{j-1} = \phi$$

for some $\mu,\theta \in \{1,\cdots,\alpha\}$ and $\mu \neq \theta$, then $\{i,\cdots,j\}$ is called a closed extendible interval of $\{B_1,\cdots,B_h\}$.

**LEMMA** 1.2 (Lemma 3.2, [6]) Let $C$ be a set, $\{A_1,\cdots,A_\alpha\}$ and $\{B_1,\cdots,B_h\}$ be partitions of $C$ defined as above. If $s$ is an integer such that $\alpha \geq s$ and $|A_\mu| \geq s$ for each $\mu \in \{1,\cdots,\alpha\}$, then $\{B_1,\cdots,B_h\}$ has at least $s-1$ closed extendible intervals.

Suppose that $G \backslash V(P)$ is not an independent set and let $W_0$ be a component of $G \backslash V(P)$ which contains at least two vertices. Let $T_1,\cdots,T_t$ be all end-blocks of $W_0$. (An end-block of $W_0$ is a block of $W_0$ which contains at most one cut-vertex of $W_0$).

I. We claim that there exists a longest path $Q_i = x_1^i \cdots x_{q_i}^i$ in each $T_i$ such that

(i) $d_{W_0}(x_1^i) \leq d_{W_0}(x_{q_i}^i)$ and $x_1^i$ is not a cut-vertex of $W_0$, and

(ii) $d_{W_0}(x_1^i)$ is as big as possible.

Let $R = y_1 \cdots y_r$ be a longest path in $T_i$ such that $d_{W_0}(y_1) \leq d_{W_0}(y_r)$.

($\alpha$). If $y_1$ is a cut-vertex of $W_0$ and $d_{T_i}(y_1) \geq 2$, then there is another longest path $y_\mu \bar{R} y_1 y_{\mu+1} R y_r$ or $y_r \bar{R} y_{\mu+1} y_1 R y_\mu$ satisfying (i) for any $y_{\mu+1} \in N_R(y_1) \setminus \{y_2\}$. Of all longest paths in $T_i$ satisfying (i), let $Q_i = x_1^i \cdots x_{q_i}^i$ be the one with the largest $d_{W_0}(x_1^i)$. ($\beta$) if $y_1$ is a cut-vertex of $W_0$ and $d_{T_i}(y_1)=1$, then $|T_i|=2$ and $R=y_1y_2$ since $T_i$ is a block. Hence $d_{W_0}(y_2)=1$ and $d_{W_0}(y_1) > 1$ because $y_1$ is a cut-vertex of $W_0$. It contradicts the assumption that $d_{W_0}(y_1) \leq d_{W_0}(y_r)$.

II. Let $d=\max\{d_{W_0}(x_i^i): i=1,\cdots,t\}$. Without loss of generality, let $d=d_{W_0}(x_1^1)$.

(i) When $d \geq 2$ and $N_{Q_1}^{-1}(x_1^1) \cap \{\text{cut-vertices of } W_0\}=\phi$, let $Z= N_{Q_1}^{-1}(x_1)$.

(ii) When $d \geq 2$ and $x_c^1$ is a vertex of $N_{Q_1}^{-1}(x_1^1) \cap \{\text{cut-vertices of } W_0\}$.

Let $$Z=[N_{Q_1}^{-1}(x_1^1) \setminus \{x_c^1\}] \cup \{x_1^2\}.$$

In both cases (i) and (ii), we have that $|Z| = |N_{Q_1}^{-1}(x_1^1)| = d_{W_0}(x_1^1) = d$, and by Lemma 1.1,

$$L_{W_0}(z,z')=L_{T_1}(z,z')$$

$$\geq \min\{d_{T_1}(x_1^1),d_{T_1}(x_{q_1}^1)\}$$

$$=d_{T_1}(x_1^1)$$

$$=d_{W_0}(x_1^1)$$

$$=d$$

for each pair of distinct vertices, $z,z'\in Z\ V(T_1)$. If $z\in Z\cap V(T_1)$ and $z'\in Z\setminus T_1$ we have that $z'=x_1^2$ and

$$L_{W_0}(z,z') \geq L_{W_0}(z,x_c^1)+L_{W_0}(x_e^1,x_1^2)$$

$$\geq L_{T_1}(z,x_c^1)$$

$$\geq d$$

By the choice of $Q_1$ and $x_1^1$, it follows that

$$d=d_{W_0}(x_1^1) \geq d_{W_0}(z)$$

for each $z\in Z$.

(iii) When $d=1$, $T_1$ is a single edge $(x_1^1,x_2^1)$. Hence, $x_1^1$ is a degree one vertex of $W_0$ and $x_2^1$ is either a cut-vertex of $W_0$ if $W_0\neq T_1$, or a degree one vertex of $W_0$ if $W_0=T_1$. If $W_0=T_1$, then let $Z=\{x_1^1,x_2^1\}$. If

$W_0 \setminus T_1 \neq \phi$, by the choice of $x_1^1$, we must have that $d_{W_0}(x_1^2) \leq d_{W_0}(x_1^1)$ and $x_1^2$ is a degree one vertex of $W_0$. Then let

$$Z=\{x_1^1, x_1^2\}.$$

Thus in either case, $d_{W_0}(z)=1$ for any $z$ Z.

So we always have that

$$|Z| = \max\{d,2\}, \qquad \cdots\cdots \qquad (1)$$

$$L_{W_0}(z,z') \geq d, \qquad \cdots\cdots \qquad (2)$$

$$d_{W_0}(z) \leq d \text{ and } d_P(z) \geq k-d \qquad \cdots\cdots \qquad (3)$$

for each pair of distinct vertices $z$ and $z'$ of $Z$. And

$$|T_1| \geq d+1 \qquad \cdots\cdots \qquad (4)$$

since $d=d_{W_0}(x_1^1)=d_{T_1}(x_1^1)$.

III. We claim that $1 \leq d \leq k-4$.

Suppose that $d \geq k-3$. Since $G$ is 4-connected, there are four intermediately disjoint paths $P_\mu=v_{i_\mu}\cdots x_\mu$ joining $T_1$ and $P$ for $\mu=1,\cdots,4$ where $\{v_{i_1},v_{i_2},v_{i_3},v_{i_4}\}$ are distinct vertices of $P$, $0 \leq i_1 < i_2 < i_3 < i_4 \leq p$, $\{x_1,x_2,x_3,x_4\}$ belong to $T_1$ and $|\{x_1,\cdots,x_4\}| =\min\{|T_1|,4\}$.

Let $R_\mu$ be a path joining $x_\mu$ and $x_{\mu+1}$ in $T_1$ such that $R_\mu$ is of length at least $d$ if $x_\mu \neq x_{\mu+1}$ (by Lemma 1.1), or $R_\mu = x_\mu$ if $x_\mu = x_{\mu+1}$. Then

$$\ell(v_{i_\mu} P v_{i_{\mu+1}}) \geq \ell(v_{i_\mu} P_\mu x_\mu R_\mu x_{\mu+1} P_{\mu+1} v_{i_{\mu+1}}) \geq d+2$$

if $x_\mu \neq x_{\mu+1}$, or $\ell(v_{i_\mu} P v_{i_{\mu+1}}) \geq \ell(v_{i_\mu} P_\mu x_\mu P_{\mu+1} v_{i_{\mu+1}}) \geq 2$

if $x_\mu = x_{\mu+1}$ since $P$ is a longest path joining $v_0$ and $v_p$.

If $|T_1| \geq 4$, then $\{x_1, x_2, x_3, x_4\}$ are a set distinct vertices and

$$\begin{aligned} \ell(P) &\geq \sum_{\mu=1}^{3} \ell(v_{i_\mu} P v_{i_{\mu+1}}) \\ &\geq 3(d+2) \\ &\geq 3k-3 \qquad \text{(by } d \geq k-3\text{)}. \end{aligned}$$

It contradicts the assumption that $\ell(P) < 3k-6$. Therefore $|T_1| \leq 3$ and some $x_i$ and $x_j$ of $\{x_1, x_2, x_3, x_4\}$ are the same vertex. However,

$$\begin{aligned} 3k-7 \geq \ell(P) &\geq \sum_{\mu=1}^{3} \ell(v_{i_\mu} P v_{i_{\mu+1}}) \\ &\geq \sum_{x_\mu \neq x_{\mu+1}} \ell(v_{i_\mu} P v_{i_{\mu+1}}) + \sum_{x_\mu = x_{\mu+1}} \ell(v_{i_\mu} P v_{i_{\mu+1}}) \\ &\geq (d+2)(|T_1|-1) + 2(4-|T_1|) \\ &= d(|T_1|-1)+6 \\ &\geq d^2+6 \qquad \text{(by (4))} \\ &\geq k^2-6k+15 \qquad \text{(by } d \geq k-3\text{)}. \end{aligned}$$

Thus $0 \geq k^2-9k+22$. But the value of $K^2-9k+22$ is always positive for any k. It leads a contradition and follows our claim.

IV. Now we wish to show the following inequality

$$\ell(P) \geq (k-d-1)(d+2) \qquad \cdots\cdots \qquad (5)$$

Let z,z' be a pair of distinct vertices of Z. We have known that

$d_P(z)$, $d_P(z') \geq k - d$ and $\ell_{W_0}(z,z') \geq d$ [by (2) and (3)]. Let

$|N_P(z) \cap N_P(z')| = \sigma(z,z')$. Since P is a longest path joining $v_0$ and

$v_p$, $N_P(z) \cup N_P(z')$ does not contain two consecutive vertices of P. Let

$\{v_{i_1},\cdots,v_{i_r}\} = N_P(z) \cup N_P(z')$. Then $[v_{i_1} P v_{1_r}] \backslash [N_P(z) \cup N_P(z')]$ contains r-1

open segments. A segment $v_{i_\theta} P v_{i_{\theta+1}}$ is called extendible with respect to

$\{z,z'\}$ if either $v_{i_\theta} \in N(z)$ and $v_{i_{\theta+1}} \in N(z')$ or $v_{i_\theta} \in N(z')$ and $v_{i_{\theta+1}} \in N(z)$.

Otherwise, it is called unextendible. It is not very hard to see that P has at least $\sigma(z,z')-1$ extendible segments with respect to $\{z,z'\}$. Since P is a longest path joining $v_0$ and $v_p$ and $L_{W_0}(z,z') \geq \alpha$, each entendible segment is of length at least d+2 and each unextendible segment is of length at least two.

(i) If there is a pair of distict vertices $\{z_1,z_2\}$ of Z such that P has $\sigma(z_1, z_2)$ or $\sigma(z\ ,z\ )-1$ extendible segments with respect to $\{z_1,z_2\}$ then one of $\{N_P(z_1),N_P(z_2)\}$ must be a subset of another one and

$\sigma(z_1, z_2) = \min\{|N_P(z_1)|, |N_P(z_2)|\} \geq k-d.$

So $\ell(P) \geq$ (total length of all extendible segments)

$$\geq (d+2)\big(\sigma(z_1,z_2)-1\big)$$

$$\geq (d+2)(k-d-1). \qquad \text{(since } \sigma(z_1,z_2) \geq k-d\text{)}$$

Thus we have established the inequality (5) in this case, and therefore we will asume that $P$ has at least $\alpha(z,z')+1$ extendible segments with respect to any pair of distinct vertices $\{z,z'\}$ of $Z$.

(ii) Case 1. $\quad d \leq \frac{k}{2}$

Let $\sigma = \max\{\sigma(z,z') \mid z,z'$ are a pair of distinct vertices of $Z\}$. Choose a pair of distinct vertices $z_1$ and $z_2$ of $Z$ such that $\sigma(z_1,z_2) = \sigma$ and let $r = |N_P(z_1) \cup N_P(z_2)|$. It is clear that

$$r+\sigma = |N_P(z_1)| + |N_P(z_2)| \geq 2(k-d) \qquad \cdots\cdots \qquad (6)$$

$$r = \geq |N_P(z_1)| \geq k-d \qquad \cdots\cdots \qquad (7)$$

Since $P$ has at least $\sigma+1$ entendible segments with respect to $\{z_1,z_2\}$, we have that

$\ell(P) \geq$ (total length of all extendible segments with respect to $\{z_1,z_2\}$ +

(total length of all unextendible segments with respect to $\{z_1,z_2\}$)

$$\geq (d+2)(\sigma+1) + 2[(r-1)-(\sigma+1)]$$

$$= 2r + \sigma d + d - 2$$

$$\geq 2[2(k-d)-\sigma] + \sigma d + d - 2 \qquad \big(\text{since } r \geq 2(k-d) - \sigma \text{ by } (6)\big)$$

$$= 4k-4d - 2\sigma + \sigma d + d - 2$$

$$= (4k-2d) -2d + (\sigma+1)(d-2)$$

$$\geq 3k - 2d + (\sigma+1)(d-2) \qquad \text{(since } d \leq \tfrac{k}{2}\text{)}$$

Thus $\quad 3k - 7 \geq \ell(P) \geq 3k - 2d + (\sigma+1)(d-2) \quad \cdots\cdots \quad (8)$

if $\sigma \geq 1$, by (8), we have that

$$3k - 7 \geq 3k -2d + 2(d-2)$$
$$= 3k - 4.$$

It is a contradiction and hence we have that $\sigma = 0$. If $d \leq 4$, by (8), we have that

$$3k - 7 \geq \ell(P) \geq 3k -2d + (d-2) \qquad \text{(since } \sigma=0\text{)}$$
$$\geq 3k - 6 \qquad \text{(since } d \leq 4\text{).}$$

It is also a contradiction and therefore we must have that $d \geq 5$. Note that $|Z| \geq d \geq 5$, let $z,z',z''$ be three distinct verticies of $Z$. By the definition of $\sigma$ and $\sigma=0$, the subsets $N_P(z)$, $N_P(z')$ and $N_P(z'')$ of $V(P)$ are pairwise disjoint. Hence

$$|N_P(z) \cup N_P(z') \cup N_P(z'')| \geq 3(k-d)$$

and $P$ has at least $3(k-d)-1$ segments each of which is of length at least two. So

$$\ell(P) \geq 2[3(k-d)-1]$$
$$= 6k - 6d - 2$$
$$\geq 3k - 2 \qquad \text{(since } d \leq \tfrac{k}{2}\text{).}$$

It contradicts that $\ell(P) \leq 3k - 7$.

(iii) Case 2. $d \geq \frac{k}{2}$ .

Let $C=E(Z,P)$

be a set and

$$\{A_z=E(z,P): \text{for each } z \in Z\}$$

and $\{B_i=E(Z,v_i): \text{for each } v_i \in V(P)\}$

be partitions of $C$. Note that $|\{A_z\}| = |Z| = d \geq k-d$ and $|A_z| = d_P(z) \geq k-d$ for any $z \in Z$ (by (3)), $|A_z \cap B_i| \leq 1$ for any $z \in Z$ and $v_i \in V(P)$. We can apply Lemma 1.2 on $C$ and these two partitions of $C$. Thus $P$ has at least $k-d-1$ extendible segments each of which is of length at least $d+2$ and therefore

$$\ell(P) \geq \text{(total length of all entendible segments)}$$
$$\geq (d+2)(k-d-1)$$

and the inequality (5) holds for all cases.

V. Since $1 \leq d \leq k-4$, the minimum value of $(d+2)(k-d-1)$ is $3k-6$ it contradicts that $\ell(P) < 3k-6$ and therefore, $G \backslash V(P)$ is an independent set.

**Part two.**

It has been shown in part one that $W=G\backslash V(P)$ is an independent set. Let $w$ W. Following [5], put $Y_0=\phi$ and for $i \geq 1$, put

$$X_i=N(Y_{i-1} \cup \{W\})$$

and $Y_i=\{v_j \in V(P): \; v_{j-1} \in X_i \text{ and } v_{j+1} \in X_i\}$ .

Thus $N(w)=X_1 \subseteq X_2 \cdots$ and $\phi=Y_0 \subseteq Y_1 \subseteq Y_2 \cdots$ .

Put $X=\bigcup_{i=1}^{\infty} X_i$ and $Y=\bigcup_{i=1}^{\infty} Y_i$. The follow lemma has been proved in [6] and will be applied in this part of the proof.

LEMMA 2.1.

(i) (direct conclusion of the definition) $Y\subseteq V(P)\setminus\{v_0,v_p\}$ and $Y=(X\cap P)^{+1}\cup(X\cap P)^{-1}$.

(ii) (Lemma 4.4. [6]) $X$ does not contain two consecutive vertices of $P$.

(iii) (Lemma 4.4. [6]) $X\cap Y=\phi$.

(iv) (Lemma 4.7. [6]) $Y\cup W$ is an indpendent set of $G$, $N(Y)\subseteq V(P)$ and $N(Y\cup\{w\})=X\subseteq V(P)$.

(v) $e(X,Y\cup\{w\})=k(|Y|+1)$ and $e(V',Y\cup\{w\})=0$ for any subset $V'$ of $V(G)\setminus X$.

Proof. We only need to prove (v). By (i) $v_0,v_p\notin Y\cup\{w\}$, it follows that $d(u)=k$ for any $u\in Y\cup\{w\}$. Since $X=N(Y\cup\{w\})$, $e(Y\cup\{w\},X)=e(Y\cup\{w\},G)=k|Y\cup\{w\}|$ and $N(Y\cup\{w\})\cap V'=\phi$ for any subset $V'$ of $V(G)\setminus X$.

Put $|X|=\chi$ and $|Y|=\psi$. Then $P\setminus X\cup Y$ is a union of at most $\chi-\psi+1$ segments of $P$. Let $S_1,\cdots,S_{t-1}$ be the segments of $P\setminus X\cup Y$ not containing $v_0$ and $v_p$. Let $S_0$ (or $S_t$) be the segment of $P\setminus X\cup Y$ containing $v_0$ (or $v_p$, respectively) if $v_0$ (or $v_p$, respectively) does not belong to $X$. Obviously, $S_0=\phi$ (or $S_t=\phi$) if $v_0\in X$ (or $v_p\in X$, respectively). It is easy

to see that $|S_i| \geq 2$ for $1 \leq i \leq t-1$ and $t=\chi-\psi$. Let $S=\bigcup_{i=0}^{t} S_i$. Here

Here $V(P)=X\cup Y\cup S$, by (i) and (iv) of Lemma 2.1.

Case 1. $S \neq \phi$.

Let $Z_i=S_i \cap (X^{+1} \cup X^{-1})$ and $Z=\bigcup_{i=0}^{t} Z_i$. We have that

LEMMA 2.2 (Lemma 4.8, [6])

$$e(Z,S) \leq (t-\lambda)(|S|-t+3)$$

where $\lambda=0$ if $S_0 \cup S_t \neq \phi$ and $\lambda=1$ if $S_0 \cup S_t=\phi$.

and

LEMMA 2.3 (Lemma 4.9, [6])

$$e(X,W\setminus\{w\}) \geq e(Z,W\setminus\{w\}).$$

Now we can prove our theorem in this case. Since

$$k\chi \geq e(X,G) \geq e(X,Z)+e(X,Y\cup\{w\})=e(X,W\setminus\{w\})$$

and

$$k|Z|=e(Z,G)=e(Z,X)+e(Z,Y\cup\{w\})+e(Z,S)+e(Z,W\setminus\{w\}),$$

we have that

$$\begin{aligned} k\chi-e(X,Y\cup\{w\})-e(X,W\setminus\{w\}) &\geq e(X,Z) \\ &=e(Z,X) \\ &=k|Z|-e(Z,S)-e(Z,W\setminus\{w\})-e(Z,Y\cup\{w\}). \end{aligned}$$

Thus

$$\begin{aligned} &k\chi-k(\psi+1)-e(X,W\setminus\{w\}) \\ &\quad \geq k|Z|-e(Z,S)-e(Z,W\setminus\{w\}) \end{aligned}$$

by (v) of Lemma 2.1. Note that $\chi-\psi=t$ and

$$e(X,W\setminus\{w\}) \geq e(Z,W\setminus\{w\})$$

(by Lemma 2.3), it follows that

$$e(Z,S) \geq -kt+k+k|Z|.$$

When $S_0 \cup S_t \neq \phi$, $|Z| \geq 2t-1$. By Lemma 2.2,

$$t(|S|-t+3) \geq -kt+k+k(2t-1).$$

Simplifying the above inequality, we have that

$$|S| \geq t-3+k. \qquad \cdots\cdots \qquad (9)$$

When $S_0 \cup S_t = \phi$, $|Z| = 2(t-1)$. By Lemma 2.2,

$$(t-1)(|S|-t+3) \geq -kt+k+2k(t-1).$$

Simplifying the above inequality, we obtain the inequality (9) again. Since $V(P)=S \cup X \cup Y$, and $t+\psi=\chi \geq |N(w)|=k$,

$$\begin{aligned} \ell(P)+1 = |V(P)| &= |S| + |X| + |Y| \\ &\geq (t-3+k)+\chi+\psi \qquad \text{(by (9))} \\ &= k+2\chi-3 \\ &\geq 3k-3 \end{aligned}$$

It contradicts that $\ell(P) < 3k-6$ and therefore the path joining $v_0$ and $v_p$ is of length at least $3k-6$ in the case of $S \neq \phi$.

Case two. $S=\phi$. In this case, we must have $p=\ell(P)$ is even and $X=\{v_{2i}: i=0,\cdots,\frac{p}{2}\}$, $Y=\{v_{2i-1}: i=1,\cdots,\frac{p}{2}\}$. Thus $\left|Y \cup \{w\}\right| = \left|X\right|$. We claim that $X$ is also an independent set and $N(X) \subseteq Y \cup \{w\}$. By (v) of Lemma 2.1, we have that

$$e(Y \cup \{w\},X)=k|Y \cup \{w\}| = k|X|.$$

Since the maximum degree of $G$ is $k$, all neighbors of every vertex of $X$ are contained in $Y \cup \{w\}$.

Moreover, by (iv) of Lemma 2.1, both $X$ and $Y \cup \{w\}$ are independent sets and

$$E(X,Y \cup \{w\})=E(X,G)=E(G,Y \cup \{w\}).$$

The connectivity of $G$ implies that $V(G)=X\cup Y\cup\{w\}$. Thus $(X,Y\ \{w\})$ is a bipartition of $G$ and $v_0,v_p$ are joined by a path of length $|V(G)|-2$.

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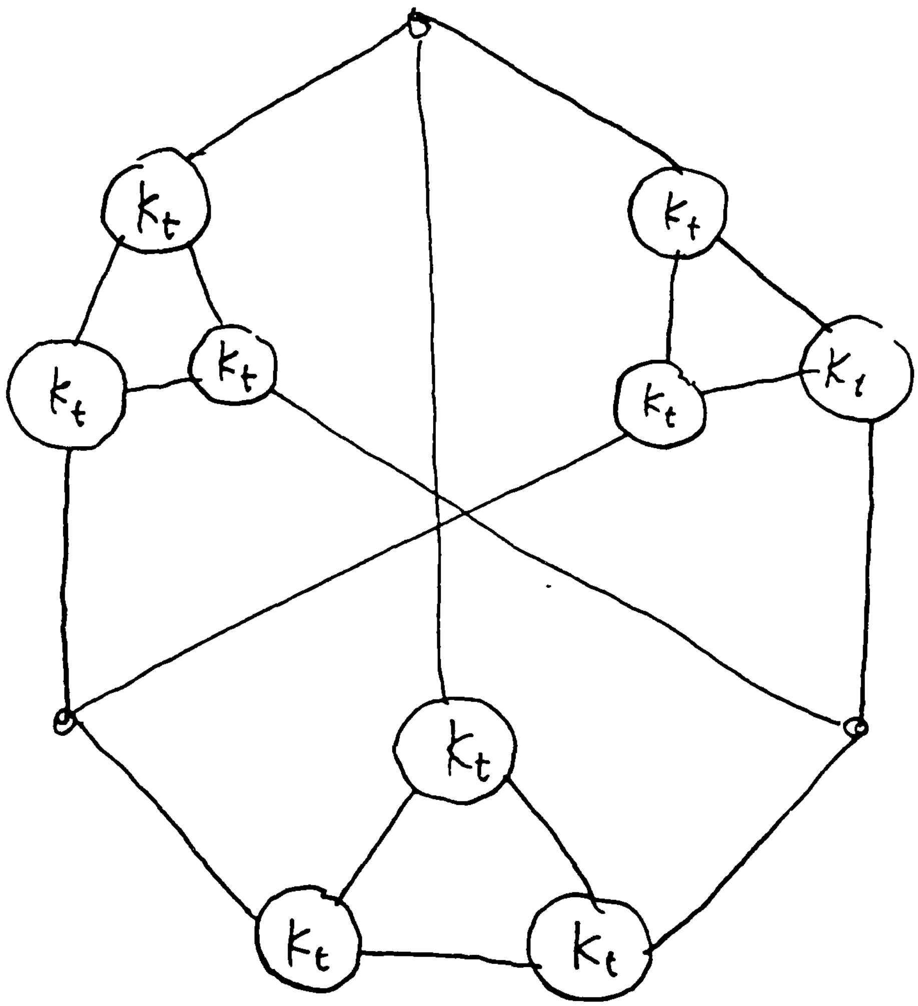


fig. 1.